ON THE YAMABE EQUATION WITH ROUGH POTENTIALS

FRANCESCA PRINARI AND NICOLA VISCIGLIA

Abstract. We study the existence of non–trivial solutions to the Yamabe equation:

$$-\Delta u + a(x) = \mu u|u|^{\frac{4}{n-2}} \ \mu > 0, x \in \Omega \subset \mathbf{R}^n \text{ with } n \geq 4,$$
$$u(x) = 0 \text{ on } \partial\Omega$$

under weak regularity assumptions on the potential $a(x)$.
More precisely in dimension $n \geq 5$ we assume that:

(1) $a(x)$ belongs to the Lorentz space $L^{\frac{n}{2},d}(\Omega)$ for some $1 \leq d < \infty$,
(2) $a(x) \leq M < \infty$ a.e. $x \in \Omega$,
(3) the set $\{x \in \Omega | a(x) < 0\}$ has positive measure,
(4) there exists $c > 0$ such that

$$\int_\Omega (|\nabla u|^2 + a(x)|u|^2) \ dx \geq c \int_\Omega |\nabla u|^2 \ dx \ \forall u \in H^1_0(\Omega).$$

In dimension $n = 4$ the hypothesis (2) above is replaced by

$$a(x) \leq 0 \ a.e. \ x \in \Omega.$$

1. Introduction

In this paper we shall look for the existence of non–trivial solutions to the following Yamabe equation:

$$-\Delta u + a(x)u = \mu u|u|^{\frac{4}{n-2}} \ \mu > 0, x \in \Omega \subset \mathbf{R}^n \text{ with } n \geq 4 \tag{1.1}$$

$$u(x) = 0 \text{ on } \partial\Omega,$$

under suitable assumptions on $a(x)$ that will be specified later.

The main strategy will be is to look at the following minimization problem:

$$S_a(\Omega) = \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{\int_\Omega (|\nabla u|^2 + a(x)|u|^2) \ dx}{\|u\|^2_{L^{2^*}}}, \tag{1.2}$$

where $\Omega \subset \mathbf{R}^n$ is an open set (eventually unbounded), $\|u\|^{2^*}_{L^{2^*}} = \int_\Omega |u|^{2^*} dx$ and $2^* = \frac{2n}{n-2}$. Let us underline that the minimization problem (1.2) is non–trivial due to the non–compactness of the Sobolev embedding:

$$H^1_0(\Omega) \subset L^{2^*}(\Omega). \tag{1.3}$$

In fact the problem (1.2) has been extensively studied in the literature, starting from the pioneering works [2] and [10], due to its obvious connection with the equation (1.1) that plays a fundamental role in Riemannian geometry (see the very complete book [1]).

The literature around this problem is too large in order to be exhaustive, howevere we want to mention at least some of these papers.

In [3] the problem (1.2) has been treated under the following assumptions: $a(x) \in L^\infty(\Omega)$, $\Omega \subset \mathbf{R}^n$ is bounded and $a(x) \leq -\epsilon < 0$ on an open subset of Ω.

In [5] it has been introduced a general approach (the concentration–compactness method) to overcome, in many minimization problems, the difficulties connected with the lack of compactness in the Sobolev embedding (1.3). In the same paper many applications of the concentration–compactness method are given, among them let us mention the problem (1.2) that is treated under suitable assumptions on $a(x)$.

Finally we want to mention [11] where the same problem is treated assuming that $a(x)$ is a function homogeneous of order -2 defined on the whole $\mathbf{R}^n$.

In this article we shall work mainly with functions $a(x)$ belonging to the Lorentz space $L^{\frac{n}{2},d}(\Omega)$ with $1 \leq d < \infty$ (for a definition of $L^{p,q}$ see [7] or section 2) without any further regularity assumption. Let us point–out that the quadratic form introduced in (1.2) is meaningful in general for every $a(x) \in L^{\frac{n}{2},d}(\Omega)$ due to the following inequality:

$$\int_\Omega |a(x)||u|^2 \, dx \leq C \int_\Omega |\nabla u|^2 \, dx \;\forall u \in H_0^1(\Omega) \tag{1.4}$$

(see theorem 1.3) where $C > 0$ depends on $a(x) \in L^{\frac{n}{2},d}(\Omega)$.

Notice that if $a(x) \equiv 0$, then the problem (1.2) is equivalent to understand whether or not the best constant in the critical Sobolev embedding (1.3) is achieved. By using the concentration–compactness method developed in [5] it is possible to show that the best constant is achieved when $\Omega \equiv \mathbf{R}^n$. On the other hand a standard rescaling argument implies that the best constant is never achieved in the case that $\Omega \neq \mathbf{R}^n$.

As it was mentioned above, in [3] the authors have shown that the situation changes when a term of the type $\int_\Omega a(x)|u|^2 \, dx$ is added to the energy $\int_\Omega |\nabla u|^2 \, dx$, provided that Ω is bounded and $a(x) \in L^\infty(\Omega)$ is negative on an open subset of Ω.

The main aim of this paper is to show that there exists a minimizer for (1.2) when $a(x)$ belongs to a class more general than the one considered in [3]. Of course for the same class of potentials $a(x)$ we can deduce the existence of non–trivial solutions to (1.1) by using a straightforward Lagrange multipliers technique.

Next we state our result in dimension $n \geq 5$.

Theorem 1.1. *Let $n \geq 5$ and let $\Omega \subset \mathbf{R}^n$ an open set (eventually unbounded). Assume that $a(x) \in L^{\frac{n}{2},d}(\Omega)$ with $d \neq \infty$, satisfies:*

$$\text{there exists } 0 \leq M < \infty \text{ such that } a(x) < M \text{ a.e. } x \in \Omega; \tag{1.5}$$

$$\text{the set } \mathcal{N} \equiv \{x \in \Omega | a(x) < 0\} \text{ has positive measure}; \tag{1.6}$$

$$\text{there exists } c > 0 \text{ such that} \tag{1.7}$$

$$\int_\Omega (|\nabla u|^2 + a(x)|u|^2) \, dx \geq c \int_\Omega |\nabla u|^2 \, dx \;\forall u \in H_0^1(\Omega).$$

Then there exists a function $v_0 \in H_0^1(\Omega)$ such that $\int_\Omega |v_0|^{2^*} dx = 1$ and

$$\inf_{H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla v|^2 + a(x)|v|^2) \, dx}{\|v\|^2_{L^{2^*}}} = \int_\Omega (|\nabla v_0|^2 + a(x)|v_0|^2) \, dx.$$

Let us notice that we do not assume the continuity of $a(x)$ and that we allow to the potential $a(x)$ to have a bounded non–negative part.

In dimension $n = 4$ we are able to give a similar result under an assumption stronger than (1.5). In fact in remark 3.2 it is explained where the proof of theorem 1.1 fails in dimension $n = 4$.

Theorem 1.2. *Let $\Omega \subset \mathbf{R}^4$ be an open set (eventually unbounded). Assume that $a(x) \in L^{2,d}(\Omega)$ with $d \neq \infty$, satisfies (1.6), (1.7) as in theorem 1.1 and*

$$a(x) \leq 0 \ a.e. \ x \in \Omega. \tag{1.8}$$

Then there exists a function $v_0 \in H_0^1(\Omega)$ such that $\int_\Omega |v_0|^{2^*} dx = 1$ and

$$\inf_{H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla v|^2 + a(x)|v|^2) \ dx}{\|v\|^2_{L^{2^*}}} = \int_\Omega (|\nabla v_0|^2 + a(x)|v_0|^2) \ dx.$$

Remark 1.1. Along the proof of theorems 1.1 and 1.2 it will be clear that the assumption (1.7) is needed only to guarantee the boundedness in $H_0^1(\Omega)$ of the minimizing sequences in (1.2). On the other hand it is easy to show that if $a(x) \in L^{\frac{n}{2}}(\Omega)$, then the boundedness of the minimizing sequences in (1.2) can be proved by using the Sobolev embedding $H_0^1(\Omega) \subset L^{2^*}(\Omega)$ without any further assumption. This implies that assumption (1.7) can be removed in theorems 1.1 and 1.2 in the case $a(x) \in L^{\frac{n}{2}}(\Omega)$. On the other hand the coercivity assumption (1.7) is natural in the literature (see for example [3], [5]).

Remark 1.2. Looking at the proof of theorems 1.1 and 1.2 it will be clear that we prove the following fact: every minimizing sequence for (1.2) is compact in $H_0^1(\Omega)$.

Remark 1.3. In general theorem 1.1 cannot be extended to potentials $a(x)$ belonging to $L^{\frac{n}{2},\infty}(\Omega)$. For instance it is possible to show that for every $\mu \in \mathbf{R}$ and $0 < R < \infty$ the value

$$H_{\mu,R} = \inf_{H_0^1(|x|<R)\setminus\{0\}} \frac{\int_{|x|<R} (|\nabla u|^2 - \mu|x|^{-2}) \ dx}{\|u\|^2_{L^{2^*}}}$$

is never achieved.

We underline that from a technical point of view the assumption $d \neq \infty$, done in the statement of theorems 1.1 and 1.2, will be relevant in order to prove the compactness of a Sobolev embedding in suitable weighted spaces. We think that this result has its own interest and we state it separately.

Theorem 1.3. *Assume $n \geq 3$ and $\Omega \subset \mathbf{R}^n$. Then for every $a(x) \in L^{\frac{n}{2},d}(\Omega)$ with $1 \leq d \leq \infty$, we have the following continuous embedding:*

$$H_0^1(\Omega) \subset L^2_{|a(x)|}(\Omega) \tag{1.9}$$

where

$$\|u\|^2_{L^2_{|a(x)|}(\Omega)} = \int_\Omega |a(x)||u|^2 \ dx. \tag{1.10}$$

If moreover $d \neq \infty$, then the embedding is compact.

Next we fix some notations useful in the sequel.

Notations.

For every $1 \leq p, q \leq \infty$ we denote by $L^{p,q}(\Omega)$ the usual Lorentz spaces (see section 2).

For every $a(x) \in L^{\frac{n}{2},d}(\Omega)$ and $\Omega \subset \mathbf{R}^n$, we shall denote by $S_a(\Omega)$ the quantity defined in (1.2).

We shall make use of the universal constant

$$S = \inf_{H_0^1(\mathbf{R}^n)\setminus\{0\}} \frac{\int_{\mathbf{R}^n} |\nabla v|^2 dx}{\|v\|_{L^{2^*}}^2}. \tag{1.11}$$

The norm in the weighted spaces $L^2_{|a(x)|}$ is the one defined in (1.10).

If $A \subset \mathbf{R}^n$ is a measurable set then we shall denote by *meas* A and χ_A the measure of A and its characteristic function respectively.

Assume that X is a topological space, then $\mathcal{C}(X)$ denotes the space of continuous and real valued functions on X.

For every $R > 0$ and $x \in \mathbf{R}^n$ we denote by $B_R(x)$ the ball of radius R and centered in x.

Given $\alpha \geq 0$ we shall denote by $0(\epsilon^\alpha)$ and $o(\epsilon^\alpha)$ any function of the variable ϵ such that:

$$\limsup_{\epsilon\to 0} |0(\epsilon^\alpha)|\epsilon^{-\alpha} < \infty \text{ and } \lim_{\epsilon\to 0} |o(\epsilon^\alpha)|\epsilon^{-\alpha} = 0,$$

respectively.

2. The Lorentz spaces $L^{p,d}(\Omega)$ and proof of theorem 1.3

In order to introduce the Lorentz spaces we associate to every measurable function its decreasing rearrangement. Assume that $g : \Omega \to \mathbf{R}$ is a measurable function defined on the measurable set $\Omega \subset \mathbf{R}^n$. At a first step we associate to the function g its distribution function:

$$m(., g) : (0, \infty] \to [0, \infty],$$

defined for every $\sigma > 0$ as follows:

$$m(\sigma, g) = meas\ \{x \in \Omega| |g(x)| > \sigma\}.$$

It is immediate to show that the distribution function defined above is monotonic decreasing.

Once the distribution function $m(\sigma, g)$ has been introduced, we can associate to g its decreasing rearrangement function g^*:

$$g^* : [0, \infty] \to [0, \infty],$$

where

$$g^*(t) = inf\{\sigma \in \mathbf{R}^+ | m(\sigma, g) < t\}.$$

We can now define the Lorentz spaces.

Definition 2.1. Assume that $1 \leq p < \infty$ and $1 \leq d < \infty$, then the measurable function $g : \Omega \to \mathbf{R}$ belongs to the space $L^{p,d}(\Omega)$ iff

$$\|g\|^d_{L^{p,d}(\Omega)} = \int_0^\infty [g^*(t)]^d t^{\frac{d}{p}-1} dt < \infty.$$

If $1 \leq p < \infty$ and $d = \infty$, then $g \in L^{p,\infty}(\Omega)$ iff

$$\|g\|_{L^{p,\infty}(\Omega)} = \sup_{t>0} g^*(t) t^{\frac{1}{p}} < \infty.$$

Next we shall describe some properties satisfied by the functions belonging to the Lorentz spaces that are important in the sequel (for the proof see [7]).

Proposition 2.1. *Assume that $1 \leq s, s', q, r \leq \infty$ and $1 \leq d_1, d_2, d_3 \leq \infty$ are such that:*

$$\frac{1}{s} = \frac{1}{q} + \frac{1}{r} \text{ and } \frac{1}{s} + \frac{1}{s'} = 1.$$

Then:

$$\int_\Omega |fg| dx \leq \|f\|_{L^{s,d_1}(\Omega)} \|g\|_{L^{s',d_2}(\Omega)} \text{ provided that } \frac{1}{d_1} + \frac{1}{d_2} \geq 1; \tag{2.1}$$

$$\|fg\|_{L^{s,d_1}(\Omega)} \leq s' \|f\|_{L^{q,d_2}(\Omega)} \|g\|_{L^{r,d_3}(\Omega)} \text{ provided that } \frac{1}{d_2} + \frac{1}{d_3} \geq \frac{1}{d_1}. \tag{2.2}$$

Next result is a well–known improved version of the classical Sobolev embedding (see [6] and [10]).

Proposition 2.2. *For every $n \geq 3$ and for every open set $\Omega \subset \mathbf{R}^n$, there exists a real constant $C = C(\Omega) > 0$ such that:*

$$\|f\|_{L^{2^*,2}(\Omega)} \leq C \|\nabla f\|_{L^2(\Omega)} \ \forall f \in H_0^1(\Omega). \tag{2.3}$$

Remark 2.1. We want to underline that (2.3) represents an improved version of the standard Sobolev embedding due to the following inclusions:

$$L^{p,d_1} \subset L^{p,p} = L^p \subset L^{p,d_2}$$

where $1 \leq d_1 \leq p \leq d_2 \leq \infty$.

We are now able to prove theorem 1.3.

Proof of thorem 1.3. First we prove the continuity of the embedding (1.9). Notice that since $L^{\frac{n}{2},d}(\Omega) \subset L^{\frac{n}{2},\infty}(\Omega)$ for every $1 \leq d \leq \infty$, we can assume $d = \infty$. By combining (2.1), (2.2) and (2.3) we get:

$$\begin{aligned} \int_\Omega |a(x)||u|^2 \, dx &\leq \|a(x)\|_{L^{\frac{n}{2},\infty}(\Omega)} \|u^2\|_{L^{\frac{n}{n-2},1}(\Omega)} \\ &\leq \frac{n}{2} \|a(x)\|_{L^{\frac{n}{2},\infty}(\Omega)} \|u\|^2_{L^{2^*,2}(\Omega)} \leq C \|a(x)\|_{L^{\frac{n}{2},\infty}(\Omega)} \|u\|^2_{H_0^1(\Omega)}. \end{aligned} \tag{2.4}$$

Next we prove the compactness of the embedding (1.9) when $d \neq \infty$.

Let $\{u_k\}_{k \in \mathbf{N}}$ be a sequence bounded in $H_0^1(\Omega)$. We can assume that up to a subsequence there exists a function $u_0 \in H_0^1(\Omega)$ such that:

$$u_k \rightharpoonup u_0 \text{ in } H_0^1(\Omega).$$

We shall show that up to a subsequence we have:

$$\lim_{k \to \infty} \int_\Omega |a(x)||u_k|^2 \, dx = \int_\Omega |a(x)||u_0|^2 \, dx, \tag{2.5}$$

and it will complete the proof. To show (2.5) let us first notice the following property:

(2.6) *for every bounded open set K such that $K \subset \Omega$, there exists $C = C(K) > 0$ such that $\|u_k\|_{H^1(K)} < C$.*

In fact it is sufficient to show that the L^2–norm of the functions $\{u_k\}_{k \in \mathbf{N}}$ are bounded on every bounded set, since the boundedness of the L^2–norm of the gradients comes from the assumption.

Due to the Hölder inequality and to the Sobolev embedding we have:

$$\|u_k\|_{L^2(K)} \leq |K|^{\frac{1}{n}} \|u_k\|_{L^{\frac{2n}{n-2}}(\Omega)} \leq C\|\nabla u_k\|_{L^2(\Omega)}$$

where $C > 0$ is a suitable constant that depends on K, then (2.6) holds.

For every $i \in \mathbf{N}$ we split the domain Ω as follows:

$$\Omega = \Omega_1^i \cup \Omega_2^i,$$

where the splitting is the one described in proposition 6.1 and corresponding to $\epsilon = \frac{1}{i}$ (see the Appendix).

Following the proof of (2.4) one can deduce that there exists a constant $C > 0$ that depends only on Ω and such that:

$$\int_{\Omega_2^i} |a(x)||u_k|^2 \, dx \leq C\|\chi_{\Omega_2^i} |a(x)|\|_{L^{\frac{n}{2},d}(\Omega)} \|\nabla u_k\|^2_{L^2(\Omega)} < \frac{C}{i} \; \forall i, k \in \mathbf{N}, \tag{2.7}$$

where we used the boundedness of the sequence $\{u_k\}_{k\in\mathbf{N}}$ in $H_0^1(\Omega)$, and the properties of Ω_2^i described in proposition 6.1.

Recall also that $\{\Omega_1^i\}_{i\in\mathbf{N}}$ is a sequence of bounded domains. We can then combine (2.6) with the compactness of the Sobolev embedding on the bounded domain in order to deduce that:

$$\|u_k - u_0\|_{L^2(\Omega_1^i)} \to 0 \text{ as } k \to \infty,$$

where $i \in \mathbf{N}$ is a fixed number. Due to propositon 6.1 we have that $|a(x)|$ is bounded on Ω_1^i, then the previous inequality implies that for every $i \in \mathbf{N}$ there exists $k(i) \in \mathbf{N}$ such that:

$$\left| \int_{\Omega_1^i} |a(x)||u_{k(i)}|^2 \, dx - \int_{\Omega_1^i} |a(x)||u_0|^2 \, dx \right| < \frac{1}{i}. \tag{2.8}$$

It is easy to show that in fact we can choose $k(i)$ in such a way that $k(i) < k(i+1)$. By combining (2.7) with (2.8), and by using a diagonalizing argument, we can conclude that up to a subsequence we have:

$$\lim_{k\to\infty} \int_\Omega |a(x)||u_k|^2 \, dx = \int_\Omega |a(x)||u_0|^2 \, dx.$$

□

3. A general approach to the problem (1.2)

Let $v_n \in H_0^1(\Omega)$ be a sequence such that:

$$\int_\Omega |v_n|^{2^*} \, dx = 1$$

and

$$\lim_{n\to\infty} \int_\Omega (|\nabla v_n|^2 + a(x)|v_n|^2) \, dx = S_a(\Omega) \tag{3.1}$$

where $a(x)$ satisfies the assumptions done in theorems 1.1 and 1.2. Notice that due to assumption (1.7) we can deduce that $\{v_n\}_{n\in\mathbf{N}}$ is bounded in $H_0^1(\Omega)$.

Moreover the weak–compactness of bounded sequences in $H_0^1(\Omega)$ and the compactness of the embedding given in theorem 1.3, imply the existence of $v_0 \in H_0^1(\Omega)$ such that up to a subsequence we have:

(1) $v_n \rightharpoonup v_0$ in $H_0^1(\Omega)$;
(2) $\lim_{n\to\infty} \int_\Omega a(x)|v_n|^2 \, dx = \int_\Omega a(x)|v_0|^2 \, dx$.

Notice that by combining (1) and (2) with (3.1) one deduce that

$$\int_\Omega (|\nabla v_0|^2 + a(x)|v_0|^2) \, dx \leq S_a(\Omega). \tag{3.2}$$

On the other hand, following the same argument as in Brézis and Nirenberg (see also [2] and [5]) and recalling (1) and (2) above, one can deduce the following implication:

$$\text{if } S_a(\Omega) < S \text{ then } \lim_{n\to\infty} \|v_n - v_0\|_{L^{2^*}(\Omega)} = 0 \tag{3.3}$$

(recall that S is defined in (1.11)). In particular if $S_a(\Omega) < S$ then $\int_\Omega |v_0|^{2^*} = 1$, and in turn it can be combined with (3.2) to deduce that the value $S_a(\Omega)$ is achieved in $H_0^1(\Omega) \setminus \{0\}$ when $S_a(\Omega) < S$.

The main purpose in next sections will be to prove that $S_a(\Omega) < S$ under the assumptions done on $a(x)$ in theorems 1.1 and 1.2.

Next we recall a basic fact proved in [3] that will be the starting point in the proof (at least in the case $n > 4$) of the inequality $S_a(\Omega) < S$.

Assume that $n \in \mathbf{N}$ is fixed. We shall denote by $u_\epsilon(x)$ the following family of rescaled functions:

$$u_\epsilon(x) = \frac{[n(n-2)\epsilon^2]^{\frac{n-2}{4}}}{(\epsilon^2 + |x|^2)^{\frac{n-2}{2}}} \; \forall x \in \mathbf{R}^n, \epsilon > 0 \tag{3.4}$$

and for every $x_0 \in \mathbf{R}^n$

$$u_{\epsilon,x_0} = u_\epsilon(x - x_0).$$

Let us recall that the functions u_ϵ defined above realize the best constant in the critical Sobolev embedding (see [9]). In fact it is possible to prove that

$$\int_{\mathbf{R}^n} |\nabla u_\epsilon|^2 \, dx = S^{\frac{n}{2}} \tag{3.5}$$

$$\int_{\mathbf{R}^n} |u_\epsilon|^{2^*} \, dx = S^{\frac{n}{2}} \tag{3.6}$$

for every $n \geq 3$.

Let us fix also a cut–off function $\eta \in C_0^\infty(|x| < 2)$ such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ in $\{|x| < 1\}$. For every $\mu > 0, x_0 \in \mathbf{R}^n$ we introduce the function

$$\eta_{\mu,x_0}(x) = \eta\left(\frac{x - x_0}{\mu}\right).$$

Next we state a basic proposition whose proof can be found in [3] (see also [2] and [8]).

Proposition 3.1. *Let $\lambda < 0$ be a fixed number and $\Omega \subset \mathbf{R}^n$ with $n \geq 5$ be an open set. For any $\mu > 0$, $x_0 \in \Omega$ there exists $c = c(n) > 0$ such that the following estimate holds:*

$$\int_{\mathbf{R}^n} (|\nabla(u_{\epsilon,x_0}\eta_{\mu,x_0})|^2 + \lambda|u_{\epsilon,x_0}\eta_{\mu,x_0}|^2) \, dx \leq S^{\frac{n}{2}} + c\lambda\epsilon^2 + 0(\epsilon^{n-2}) \tag{3.7}$$

(here $0(\epsilon^{n-2})$ depends on $\mu > 0$).
Moreover for every $n \geq 4$ and for every $\mu > 0$ we have

$$\|u_{\epsilon,x_0}\eta_{\mu,x_0}\|^{2^*}_{L^{2^*}} = S^{\frac{n}{2}} + 0(\epsilon^n) \tag{3.8}$$

(here $0(\epsilon^n)$ depends on μ).

Remark 3.1. If $n = 4$ then the following asymptotic behaviour is given in [3]:

$$\int_{\mathbf{R}^4} (|\nabla(u_{\epsilon,x_0}\eta_{\mu,x_0})|^2 + \lambda|u_{\epsilon,x_0}\eta_{\mu,x_0}|^2)\, dx \leq S^2 + c\lambda\epsilon^2|\ln\epsilon| + 0(\epsilon^2). \tag{3.9}$$

In fact along the proof of theorem 1.2 (more precisely in the proof of lemma 5.1) we shall need a slightly refined version of this estimate.

Remark 3.2. Notice that $u_\epsilon(x) \in L^2(\mathbf{R}^n)$ for every $n > 4$ and $u_\epsilon(x) \notin L^2(\mathbf{R}^4)$ in the case $n = 4$. In next sections it will be clear that this is the main reason why the dimension $n = 4$ will be treated in a different way compared with the dimensions $n > 4$.

4. Proof of theorem 1.1

In this section the functions u_{ϵ,x_0} and η_{μ,x_0} are the ones introduced in section 3.

Let us recall also that in order to prove theorem 1.1 it is sufficient to prove the following lemma (see section 3).

Lemma 4.1. *Assume $n \geq 5$ and $\Omega \subset \mathbf{R}^n$. If $a(x)$ satisfies the assumptions of theorem 1.1 then $S_a(\Omega) < S$.*

Proof. Since now on we assume that a representative of the function $a(x)$ has been fixed and we shall not consider $a(x)$ as a class of functions that are equivalent modulo zero measure sets. This will allow us to consider the pointwise value $a(x)$ for every fixed $x \in \Omega$.

First of all we notice that we can assume $a(x) \in L^\infty(\Omega) \cap L^{\frac{n}{2},d}(\Omega)$.

In fact it is easy to show that assumption (1.6) implies that there exists $N_0 \in \mathbf{N}$ such that

$$meas\ \{x \in \Omega | -N_0 < a(x) < 0\} > 0.$$

In particular the potential

$$\tilde{a}(x) = Max\{a(x), -N_0\}$$

satisfies the assumptions of theorem 1.1 and moreover $\tilde{a}(x) \in L^\infty(\Omega) \cap L^{\frac{n}{2},d}(\Omega)$ (notice that this is stronger than (1.5)). By combining this fact with the following trivial inequality:

$$S_a(\Omega) \leq S_{\tilde{a}}(\Omega),$$

one deduce that it is not restrictive to assume $a(x) \in L^\infty(\Omega) \cap L^{\frac{n}{2},d}(\Omega)$. In particular we can assume that $a(x) \in L^{\frac{n}{2}}_{loc}(\Omega)$.

We are then in position to use the Lebesgue derivation theorem in order to deduce that (see [4] for a proof):

$$\lim_{\epsilon \to 0} \epsilon^{-n} \int_{B_\epsilon(\bar{x})} |a(x) - a(\bar{x})|^{\frac{n}{2}}\, dx = 0 \ a.e.\ \bar{x} \in \Omega,$$

that due to assumption (1.6) implies the existence of $x_0 \in \Omega$ such that

$$\lim_{\epsilon \to 0} \epsilon^{-n} \int_{B_\epsilon(x_0)} |a(x) - a(x_0)|^{\frac{n}{2}}\, dx = 0 \text{ and } -\infty < a(x_0) < 0. \tag{4.1}$$

Due again to the the definition of $S_a(\Omega)$ it is easy to verify that the following inequality holds:

$$S_a(\Omega) \leq S_{Max\{a(x),a(x_0)\}}(\Omega),$$

and it implies clearly that it not restrictive to assume that:

$$a(x_0) \leq a(x) \leq M \ a.e. \ x \in \Omega \tag{4.2}$$

(here we have used (1.5) in the r.h.s. inequality).

Next we notice that the following identity holds trivially:

$$\int_\Omega (|\nabla(u_{\epsilon,x_0}\eta_{\mu,x_0})|^2 + a(x)|u_{\epsilon,x_0}\eta_{\mu,x_0}|^2) \ dx \tag{4.3}$$

$$= \int_\Omega (|\nabla(u_{\epsilon,x_0}\eta_{\mu,x_0})|^2 + a(x_0)|u_{\epsilon,x_0}\eta_{\mu,x_0}|^2) \ dx + \int_\Omega (-a(x_0) + a(x))|u_{\epsilon,x_0}\eta_{\mu,x_0}|^2 \ dx$$

where $\mu > 0$ is choosen small enough in such a way that $u_{\epsilon,x_0}\eta_{\mu,x_0} \in H^1_0(\Omega)$.

By using the Hölder inequality we get:

$$\left| \int_\Omega (-a(x_0) + a(x))|u_{\epsilon,x_0}\eta_{\mu,x_0}|^2 \ dx \right| \tag{4.4}$$

$$\leq \int_{\Omega \cap B_{\epsilon R}(x_0)} |a(x_0) - a(x)||u_{\epsilon,x_0}\eta_{\mu,x_0}|^2 \ dx$$

$$+ \int_{\Omega \cap \{|x-x_0|>\epsilon R\}} |a(x_0) - a(x)||u_{\epsilon,x_0}\eta_{\mu,x_0}|^2 \ dx$$

$$\leq \|u_\epsilon\|^2_{L^{2^*}(\mathbf{R}^n)} \left(\int_{\Omega \cap B_{\epsilon R}(x_0)} |a(x_0) - a(x)|^{\frac{n}{2}} \ dx \right)^{\frac{2}{n}}$$

$$+2Max \ \{|a(x_0)|, M\} \int_{|x|>\epsilon R} |u_\epsilon|^2 \ dx$$

where we have used (4.2) and $R > 0$ is a real number that we are going to fix.

In fact we choose $R > 0$ large enough in such a way that the following condition holds:

$$\int_{|x|>\epsilon R} |u_\epsilon|^2 \ dx = \frac{[n(n-2)]^{\frac{n-2}{2}}}{\epsilon^{n-2}} \int_{|x|>\epsilon R} \frac{dx}{[1 + \epsilon^{-2}|x|^2]^{n-2}} \tag{4.5}$$

$$= [n(n-2)]^{\frac{n-2}{2}} \epsilon^2 \int_{|x|>R} \frac{dx}{[1 + |x|^2]^{n-2}} < \frac{c \ |a(x_0)|}{4Max \ \{|a(x_0)|, M\}} \epsilon^2$$

where $c > 0$ is the same constant that appears in (3.7).

On the other hand due to (4.1) we deduce that

$$\lim_{\epsilon \longrightarrow 0} R^{-2}\epsilon^{-2} \left(\int_{B_{\epsilon R}(x_0)} |a(x_0) - a(x)|^{\frac{n}{2}} \ dx \right)^{\frac{2}{n}} = 0. \tag{4.6}$$

By combining (4.3), (4.4), (4.5), (4.6) with (3.7) we get:

$$\int_\Omega (|\nabla(u_{\epsilon,x_0}\eta_{\mu,x_0})|^2 + a(x)|u_{\epsilon,x_0}\eta_{\mu,x_0}|^2) \ dx$$

$$\leq S^{\frac{n}{2}} + c \ a(x_0)\epsilon^2 + \frac{c}{2} \ |a(x_0)|\epsilon^2 + S^{\frac{n-2}{2}} R^2 \epsilon^2 o(1) + 0(\epsilon^{n-2})$$

$$= S^{\frac{n}{2}} + \frac{c}{2} \ a(x_0)\epsilon^2 + o(\epsilon^2).$$

By using now (3.8) we deduce that for $\epsilon > 0$ small enough the following chain of inequalities holds:

$$S_a(\Omega) \leq \frac{\int_\Omega (|\nabla(u_{\epsilon,x_0}\eta_{\mu,x_0})|^2 + a(x)|u_{\epsilon,x_0}\eta_{\mu,x_0}|^2)\ dx}{\|u_{\epsilon,x_0}\eta_{\mu,x_0}\|^2_{L^{2^*}(\Omega)}}$$
$$\leq \frac{S^{\frac{n}{2}} + \frac{c}{2}\ a(x_0)\epsilon^2 + o(\epsilon^2)}{(S^{\frac{n}{2}} + 0(\epsilon^n))^{\frac{2}{2^*}}} < S,$$

where at the last step we have used that $c\ a(x_0) < 0$.

□

5. Proof of theorem 1.2

Notice that for $n = 4$ the functions $u_\epsilon(x)$ (that have been introduced in section 3) become:

$$u_\epsilon(x) = \frac{\sqrt{8}\ \epsilon}{\epsilon^2 + |x|^2}\ \forall x \in \mathbf{R}^4\ \forall \epsilon > 0.$$

We shall also need

$$u_{\epsilon,x_0} = u_\epsilon(x - x_0)$$

and

$$\eta_{\mu,x_0} = \eta\left(\frac{x - x_0}{\mu}\right)$$

where $x_0 \in \mathbf{R}^4$, $\mu > 0$ and η is a cut–off function belonging to $C_0^\infty(|x| < 2)$ such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ in $\{|x| < 1\}$.

Let us recall also that in dimension $n = 4$ the identity (3.8) becomes:

$$\int_{\mathbf{R}^4} |u_{\epsilon,x_0}\eta_{\mu,x_0}|^4\ dx = S^2 + 0(\epsilon^4) \tag{5.1}$$

where $0(\epsilon^4)$ depends on $\mu > 0$.

Next we prove a lemma that is sufficient to conclude the proof of theorem 1.2 (see section 3).

Lemma 5.1. *Assume that $\Omega \subset \mathbf{R}^4$ is an open set and $a(x)$ satisfies the assumptions of theorem 1.2 then $S_a(\Omega) < S$.*

Proof. As in the proof of lemma 4.1 we assume that a representative of the function $a(x)$ has been fixed. This will allow us to consider the pointwise value $a(x)$ for every fixed $x \in \Omega$.

Arguing as in the first part of the proof of lemma 4.1 we can assume the existence of $x_0 \in \Omega$ such that:

$$\lim_{\epsilon \to 0} \epsilon^{-4} \int_{B_\epsilon(x_0)} |a(x) - a(x_0)|^2\ dx = 0 \text{ with } -\infty < a(x_0) < 0 \tag{5.2}$$

and

$$a(x_0) \leq a(x) \leq 0\ a.e.\ x \in \Omega \tag{5.3}$$

(here we have used (1.8) in the r.h.s inequality).

Since now on we fix $\mu > 0$ such that

$$supp\ \eta_{\mu,x_0} \subset \Omega$$

(in fact this condition is sufficient to deduce that $u_{\epsilon,x_0}\eta_{\mu,x_0} \in H_0^1(\Omega)$).

Let us write the following trivial identity:

$$\int_\Omega (|\nabla(u_{\epsilon,x_0}\eta_{\mu,x_0})|^2 + a(x)|u_{\epsilon,x_0}\eta_{\mu,x_0}|^2)\ dx = I_\epsilon^\mu + II_\epsilon^\mu \tag{5.4}$$

where

$$I_\epsilon^\mu = \int_\Omega (|\nabla(u_{\epsilon,x_0}\eta_{\mu,x_0})|^2 + a(x_0)|u_{\epsilon,x_0}\eta_{\mu,x_0}|^2)\ dx$$

$$II_\epsilon^\mu = \int_\Omega (-a(x_0)+a(x))|u_{\epsilon,x_0}\eta_{\mu,x_0}|^2\ dx.$$

Estimate for I_ϵ^μ

Notice that we have:

$$\begin{aligned}\int_\Omega |\nabla(u_{\epsilon,x_0}\eta_{\mu,x_0})|^2\ dx &= \int_\Omega |\eta_{\mu,x_0}\nabla u_{\epsilon,x_0} + u_{\epsilon,x_0}\nabla\eta_{\mu,x_0}|^2\ dx\\ &\le \int_\Omega |\eta_{\mu,x_0}\nabla u_{\epsilon,x_0}|^2\ dx + \int_\Omega |u_{\epsilon,x_0}\nabla\eta_{\mu,x_0}|^2\ dx\\ &\quad +2\int_\Omega \eta_{\mu,x_0}|\nabla\eta_{\mu,x_0}||\nabla u_{\epsilon,x_0}||u_{\epsilon,x_0}|\ dx\\ \le 32\epsilon^2\int_{|x|<2\mu}\frac{|x|^2dx}{(\epsilon^2+|x|^2)^4} &+ \frac{8\epsilon^2}{\mu^2}\|\nabla\eta\|^2_{L^\infty(\mathbf{R}^4)}\int_{\mu<|x|<2\mu}\frac{dx}{(\epsilon^2+|x|^2)^2}\\ &+\frac{32\ \epsilon^2}{\mu}\|\nabla\eta\|_{L^\infty(\mathbf{R}^4)}\int_{\mu<|x|<2\mu}\frac{|x|dx}{(\epsilon^2+|x|^2)^3}\end{aligned}$$

and then with elementary computations

$$\begin{aligned}\int_\Omega |\nabla(u_{\epsilon,x_0}\eta_{\mu,x_0})|^2\ dx &\le \int_{\mathbf{R}^4}|\nabla u_\epsilon|^2\ dx - 32\int_{|x|>2\mu}\frac{\epsilon^2|x|^2dx}{(\epsilon^2+|x|^2)^4}\\ &+\left(32\omega_3\|\nabla\eta\|^2_{L^\infty(\mathbf{R}^n)} + 256\omega_3\|\nabla\eta\|_{L^\infty(\mathbf{R}^n)}\right)\frac{\epsilon^2}{\mu^2},\end{aligned}\tag{5.5}$$

where ω_3 denotes the Haussdorf measure of the sphere $\mathcal{S}^3$.

Notice that due to (3.5) the inequality (5.5) implies

$$\int_\Omega |\nabla(u_{\epsilon,x_0}\eta_{\mu,x_0})|^2\ dx \le S^2 + C\frac{\epsilon^2}{\mu^2},\tag{5.6}$$

where $C>0$ is an universal constant.

On the other hand

$$\begin{aligned}\int_\Omega |u_{\epsilon,x_0}\eta_{\mu,x_0}|^2\ dx &\ge 8\epsilon^2\int_{|x|<\epsilon R}\frac{dx}{(\epsilon^2+|x|^2)^2}\\ &+8\epsilon^2\int_{\epsilon R<|x|<\mu}\frac{dx}{(\epsilon^2+|x|^2)^2}\end{aligned}\tag{5.7}$$

where $R>0$ is a number that we shall fix later and $\epsilon<\frac{\mu}{R}$ (recall that μ has been fixed above).

By combining (5.6) with (5.7) and recalling that $a(x_0)<0$ we get:

$$\begin{aligned}I_\epsilon^\mu &\le S^2 + C\frac{\epsilon^2}{\mu^2}\\ &+8a(x_0)\epsilon^2\int_{|x|<\epsilon R}\frac{dx}{(\epsilon^2+|x|^2)^2} + 8a(x_0)\epsilon^2\int_{\epsilon R<|x|<\mu}\frac{dx}{(\epsilon^2+|x|^2)^2}\end{aligned}\tag{5.8}$$

where $C > 0$ is an universal constant.

Estimate for II_ϵ^μ

The Hölder inequality implies:

$$\left|\int_\Omega (-a(x_0) + a(x))|u_{\epsilon,x_0}\eta_{\mu,x_0}|^2 \, dx\right| \tag{5.9}$$

$$\leq \|u_\epsilon\|^2_{L^4(\mathbf{R}^4)} \left(\int_{B_{\epsilon R}(x_0)} |a(x_0) - a(x)|^2 \, dx\right)^{\frac{1}{2}} + |a(x_0)| \int_{\epsilon R<|x|<2\mu} |u_\epsilon|^2 \, dx,$$

where we have used (5.3) to deduce $|a(x) - a(x_0)| \leq |a(x_0)|$, while $R > 0$ is a constant that we shall fix later.

Notice that (5.2) implies

$$\lim_{\epsilon\to 0} (\epsilon R)^{-2} \left(\int_{B_{\epsilon R}(x_0)} |a(x_0) - a(x)|^2 dx\right)^{\frac{1}{2}} = 0, \tag{5.10}$$

while the definition of u_ϵ gives

$$\int_{\epsilon R<|x|<2\mu} |u_\epsilon|^2 dx = 8\epsilon^2 \int_{\epsilon R<|x|<2\mu} \frac{dx}{(\epsilon^2 + |x|^2)^2}. \tag{5.11}$$

By combining (3.6), (5.9), (5.10) and (5.11) we get:

$$|II_\epsilon^\mu| \leq SR^2\epsilon^2 \, o(1) + 8\epsilon^2 \, |a(x_0)| \int_{\epsilon R<|x|<2\mu} \frac{dx}{(\epsilon^2 + |x|^2)^2}. \tag{5.12}$$

Due to (5.4), (5.8) and (5.12) we finally get:

$$\int_\Omega (|\nabla(u_{\epsilon,x_0}\eta_{\mu,x_0})|^2 + a(x)|u_{\epsilon,x_0}\eta_{\mu,x_0}|^2) \, dx \leq S^2 + C\frac{\epsilon^2}{\mu^2}$$

$$+8a(x_0)\epsilon^2 \int_{|x|<\epsilon R} \frac{dx}{(\epsilon^2 + |x|^2)^2} + 8|a(x_0)|\epsilon^2 \int_{\mu<|x|<2\mu} \frac{dx}{(\epsilon^2 + |x|^2)^2} + R^2 o(\epsilon^2)$$

$$\leq S^2 + C\frac{\epsilon^2}{\mu^2} + 8a(x_0)\epsilon^2 \int_{|x|<R} \frac{dx}{(1 + |x|^2)^2} + 8\omega_3|a(x_0)|\epsilon^2 \ln 2 + R^2 o(\epsilon^2)$$

where $C > 0$ is an universal constant, ω_3 is the measure of the sphere $\mathcal{S}^3$ and $R > 0$ is a number to be fixed later.

Then we have proved the following estimate:

$$\int_\Omega (|\nabla(u_{\epsilon,x_0}\eta_{\mu,x_0})|^2 + a(x)|u_{\epsilon,x_0}\eta_{\mu,x_0}|^2) \, dx \tag{5.13}$$

$$\leq S^2 + \left(8\phi(R)a(x_0) + \frac{C}{\mu^2} + 8\omega_3|a(x_0)| \ln 2\right)\epsilon^2 + R^2 o(\epsilon^2),$$

where

$$\phi(R) = \int_{|x|<R} \frac{dx}{(1 + |x|^2)^2}$$

and hence

$$\lim_{R\to\infty} \phi(R) = \infty. \tag{5.14}$$

Due to (5.1) and (5.13) we deduce that for $\epsilon > 0$ small enough we get:

$$S_a(\Omega) \leq \frac{\int_\Omega (|\nabla(u_{\epsilon,x_0}\eta_{\mu,x_0})|^2 + a(x)|u_{\epsilon,x_0}\eta_{\mu,x_0}|^2)\ dx}{\|u_{\epsilon,x_0}\eta_{\mu,x_0}\|^2_{L^{2^*}(\Omega)}} \tag{5.15}$$

$$\leq \frac{S^2 + \left(8\phi(R)a(x_0) + \frac{C}{\mu^2} + 8\omega_3|a(x_0)|\ln 2\right)\epsilon^2 + R^2 o(\epsilon^2)}{(S^2 + 0(\epsilon^4))^{\frac{1}{2}}} < S$$

where we have used at the last step that $a(x_0) < 0$ and we are assuming that $R > 0$ is large enough in order to guarantee that $8\phi(R)a(x_0) + \frac{C}{\mu^2} + 8\omega_3|a(x_0)|\ln 2 < 0$ (note that it is possible due to (5.14)).

□

6. Appendix

In order to make this article self–contained we give the proof of a proposition contained in [13]. We recall also that next result has been fundamental along the proof of theorem 1.3.

Proposition 6.1. *Let $n \geq 1$ and $\Omega \subset \mathbf{R}^n$ an open set. Assume that $a(x) \in L^{p,d}(\Omega)$ for $1 \leq p, d < \infty$. Then for any $\epsilon > 0$ there exist two measurable sets $\Omega_1^\epsilon, \Omega_2^\epsilon$ such that:*

$$\Omega_1^\epsilon \cup \Omega_2^\epsilon = \Omega, \Omega_1^\epsilon \cap \Omega_2^\epsilon = \emptyset, \Omega_1^\epsilon \text{ is bounded}$$

and

$$a(x)\chi_{\Omega_1^\epsilon} \in L^\infty(\Omega),\ \|a(x)\chi_{\Omega_2^\epsilon}\|_{L^{p,d}(\Omega)} < \epsilon.$$

In next lemma the function f^* associated to a function f is the one defined in section 2.

Lemma 6.1. *Assume that $f_k : \Omega \to \mathbf{R}$ is a sequence of functions such that:*

$$f_k(x) \geq 0 \text{ a.e. } x \in \Omega,\ f_1 \in L^{p,d}(\Omega) \text{ for suitable } 1 \leq p < \infty, 1 \leq d < \infty,$$

$$0 \leq f_{k+1}(x) \leq f_k(x) \text{ a.e. } x \in \Omega\ \forall k \in \mathbf{N}, \tag{6.1}$$

$$\text{and } \lim_{k\to\infty} f_k(x) = 0 \text{ a.e. } x \in \Omega. \tag{6.2}$$

Then

$$f^*_{k+1}(t) \leq f^*_k(t)\ \forall t \in \mathbf{R}^+\ \forall k \in \mathbf{N} \tag{6.3}$$

and

$$\lim_{k\to\infty} f^*_k(t) = 0\ \forall t \in \mathbf{R}^+. \tag{6.4}$$

Proof. The assumption $0 \leq f_{k+1} \leq f_k$ implies that:

$$m(\sigma, f_{k+1}) \leq m(\sigma, f_k),$$

where $m(\sigma, g)$ is defined as in section 2 for every measurable function g. Due to this inequality and to the definition of f^*_k it is easy to deduce that $f^*_{k+1}(t) \leq f^*_k(t)$ and hence (6.3) is proved.

Moreover due to (6.1) and (6.2), we have that for every fixed $\sigma > 0$, the sets

$$\mathcal{A}^\sigma_k \equiv \{x \in \mathbf{R}^n | f_k(x) > \sigma\},$$

satisfy the following properties:

$$\mathcal{A}_{k+1}^{\sigma} \subset \mathcal{A}_k^{\sigma} \ \forall k \in \mathbf{N} \ \forall \sigma \in \mathbf{R}^+ \text{ and } meas(\cap_{k\in\mathbf{N}} \mathcal{A}_k^{\sigma}) = 0 \ \forall \sigma \in \mathbf{R}^+. \tag{6.5}$$

On the other hand, since $f_1 \in L^{p,d}(\Omega^n)$ with $p, d \neq \infty$, it is easy to deduce that $meas(\mathcal{A}_1^{\sigma}) < \infty$ for every $\sigma > 0$. By combining this fact with (6.5), we can deduce that:

$$\lim_{k\to\infty} m(\sigma, f_k) = \lim_{k\to\infty} meas(\mathcal{A}_k^{\sigma}) = 0 \ \forall \sigma > 0.$$

In particular for every $\epsilon > 0$ there exists $k(\epsilon) \in \mathbf{N}$ such that

$$m(\epsilon, f_{k(\epsilon)}) < \epsilon.$$

This inequality implies that if $t > 0$ is a fixed number, then

$$f_{k(\epsilon)}^*(t) \equiv inf\{\sigma | m(\sigma, f_{k(\epsilon)}) < t\} < \epsilon,$$

provided that $0 < \epsilon < t$.

This estimate, combined with the monotonicity of $\{f_k^*(t)\}_{k\in\mathbf{N}}$ (see (6.3)), implies easily (6.4).

□

Proof of proposition 6.1. Let us introduce the following sets:

$$\Omega_k \equiv \{x \in \Omega | |a(x)| < k \text{ and } |x| < k\},$$

where $k \in \mathbf{N}$.

It is easy to show that the sequence of sets $\{\Omega_k\}_{k\in\mathbf{N}}$ satisfy the following conditions:

$$\Omega_k \subset \Omega_{k+1}, |\Omega_k| < \infty \ \forall k \in \mathbf{N} \text{ and } a(x)\chi_{\Omega_k} \in L^{\infty}(\Omega).$$

It is then sufficient to prove that for every fixed $\epsilon > 0$, there exists of a suitable $k_0(\epsilon) \in \mathbf{N}$ such that

$$\|a(x)\chi_{\Omega\setminus\Omega_{k_0(\epsilon)}}\|_{L^{p,d}(\Omega)} < \epsilon, \tag{6.6}$$

in order to conclude that the sets

$$\Omega_1^{\epsilon} = \Omega_{k_0(\epsilon)} \text{ and } \Omega_2^{\epsilon} = \Omega \setminus \Omega_{k_0(\epsilon)},$$

satisfy the desired properties.

In order to show (6.6) let us introduce the sequence of functions

$$a_k^*(t) : \mathbf{R}^+ \to \mathbf{R}^+$$

where:

$$a_k^*(t) = (|a(x)|\chi_{\Omega\setminus\Omega_k})^*(t) \ \forall k \in \mathbf{N},$$

and as usual f^* denotes the decreasing rearrangement of the function f.

If $|a|^*(t)$ denotes the rearranged function associated to $|a|$, then by using lemma 6.1 we deduce that the sequence $\{a_k^*(t)\}_{k\in\mathbf{N}}$ satisfies the following inequalities:

$$|a|^*(t) \geq a_k^*(t) \geq 0 \ \forall t > 0 \text{ and } \forall k \in \mathbf{N}, \tag{6.7}$$

$$\lim_{k\to\infty} a_k^*(t) = 0 \ \forall t \in \mathbf{R}^+. \tag{6.8}$$

In particular, since $a(x) \in L^{p,d}(\Omega)$, we have

$$|a_k^*(t)|^d t^{\frac{d}{p}-1} \leq (|a|^*(t))^d t^{\frac{d}{p}-1} \in L^1(\mathbf{R}^+)$$

that can be combined with the dominated convergence theorem and with (6.8) in order to deduce that

$$\lim_{k\to\infty} \|a(x)\chi_{\Omega\setminus\Omega_k}\|^d_{L^{p,d}(\Omega)} = \lim_{k\to\infty}\int_0^\infty |a_k^*(t)|^d t^{\frac{d}{p}-1}dt = 0.$$

The proof is complete.

□

References

[1] Aubin T. *Some nonlinear problems in Riemannian geometry*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.

[2] Aubin T. *Equations différentielles non linéaires et porbléme de Yamabe concernant la courbure scalaire,* J.Math.Pures et Appl., vol. 55, 1976, 269–293.

[3] Brézis, H. and Nirenberg, L. *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., vol. 36 (4), 1983, 437–477.

[4] Evans, L. and Gariepy R.F. *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, 1992.

[5] Lions, P.L.*The concentration compactness principle in the calculus of variations. The limit case. I*, Rev. Mat. Iberoamericana, vol.1 (1), 1985, 145–201.

[6] Malý, J. and Pick, L. *An elementary proof of sharp Sobolev embeddings*, Proc. Amer. Math. Soc., vol. 130 (2), 2002, 555–563.

[7] O'Neil, R. *Convolution operators and $L(p, q)$ spaces*, Duke Math. J., vol. 30, 1963, 129–142.

[8] Struwe, M. *Variational methods*, Applications to nonlinear partial differential equations and Hamiltonian systems, Springer-Verlag, Berlin, 1990.

[9] Talenti, G. *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl., vol. 110, 1976, 353–372.

[10] Tartar, L. *Embedding theorems of Sobolev spaces into Lorentz spaces*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) vol. 1 (3), 1998, 479–500.

[11] Terracini, S. *On positive entire solutions to a class of equations with a singular coefficient and critical exponent*, Adv. Differential Equations, vol. 1 (2), 1996, 241–264.

[12] Trudinger, N. *Remarks concerning the conformal deformation of Riemannnian structures on compact manifolds*, Ann. Sc. Norm. Sup. Pisa, vol. 22, 1968, 265–274.

[13] Visciglia, N. *A note about the generalized Hardy-Sobolev inequality with potential in* $L^{p,d}(\mathbf{R}^n)$, Calc. Var. Partial Differential Equations, vol. 24 (2), 2005, 167–184.

Francesca Prinari, Dipartimento di Matematica Università di Lecce, Via Provinciale Lecce–Arnesano, 73100 Lecce, Italy

E-mail address: francesca.prinari@unile.it

Nicola Visciglia, Dipartimento di Matematica Università di Pisa, Largo B. Pontecorvo 5, 56100 Pisa, Italy

E-mail address: viscigli@dm.unipi.it